It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

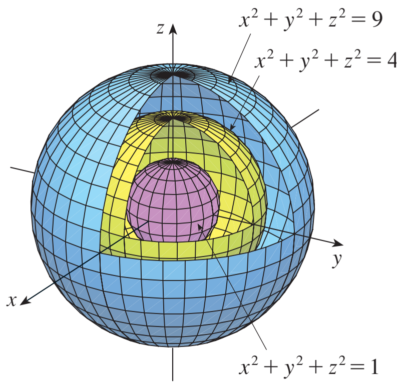


FIGURE 20

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geqslant 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 20.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed. ■

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$\boxed{3} \qquad C = f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

14.1 EXERCISES

1. In Example 2 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1.
   (a) What is the value of $f(-15, 40)$? What is its meaning?
   (b) Describe in words the meaning of the question "For what value of v is $f(-20, v) = -30$?" Then answer the question.
   (c) Describe in words the meaning of the question "For what value of T is $f(T, 20) = -49$?" Then answer the question.
   (d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
   (e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

2. The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h, so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

TABLE 3 Apparent temperature as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F) T \ h	20	30	40	50	60	70
80	77	78	79	81	82	83
85	82	84	86	88	90	93
90	87	90	93	96	100	106
95	93	96	101	107	114	124
100	99	104	110	120	132	144

(a) What is the value of $f(95, 70)$? What is its meaning?
(b) For what value of h is $f(90, h) = 100$?
(c) For what value of T is $f(T, 50) = 88$?
(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h.

3. Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 3 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

4. The wind-chill index W discussed in Example 2 has been modeled by the following function:

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Check to see how closely this model agrees with the values in Table 1 for a few values of T and v.

5. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.
(a) What is the value of $f(40, 15)$? What is its meaning?
(b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
(c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

TABLE 4

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

6. Let $f(x, y) = \ln(x + y - 1)$.
(a) Evaluate $f(1, 1)$. (b) Evaluate $f(e, 1)$.
(c) Find and sketch the domain of f.
(d) Find the range of f.

7. Let $f(x, y) = x^2e^{3xy}$.
(a) Evaluate $f(2, 0)$. (b) Find the domain of f.
(c) Find the range of f.

8. Find and sketch the domain of the function $f(x, y) = \sqrt{1 + x - y^2}$. What is the range of f?

9. Let $f(x, y, z) = e^{\sqrt{z-x^2-y^2}}$.
(a) Evaluate $f(2, -1, 6)$. (b) Find the domain of f.
(c) Find the range of f.

10. Let $g(x, y, z) = \ln(25 - x^2 - y^2 - z^2)$.
(a) Evaluate $g(2, -2, 4)$. (b) Find the domain of g.
(c) Find the range of g.

11–20 Find and sketch the domain of the function.

11. $f(x, y) = \sqrt{x + y}$

12. $f(x, y) = \sqrt{xy}$

13. $f(x, y) = \ln(9 - x^2 - 9y^2)$

14. $f(x, y) = \sqrt{y - x}\ln(y + x)$

15. $f(x, y) = \sqrt{1 - x^2} - \sqrt{1 - y^2}$

16. $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

17. $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$

18. $f(x, y) = \arcsin(x^2 + y^2 - 2)$

19. $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

20. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

21–29 Sketch the graph of the function.

21. $f(x, y) = 3$

22. $f(x, y) = y$

23. $f(x, y) = 10 - 4x - 5y$

24. $f(x, y) = \cos x$

25. $f(x, y) = y^2 + 1$

26. $f(x, y) = 3 - x^2 - y^2$

27. $f(x, y) = 4x^2 + y^2 + 1$

28. $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

29. $f(x, y) = \sqrt{x^2 + y^2}$

30. Match the function with its graph (labeled I–VI).Give reasons for your choices.

(a) $f(x, y) = |x| + |y|$

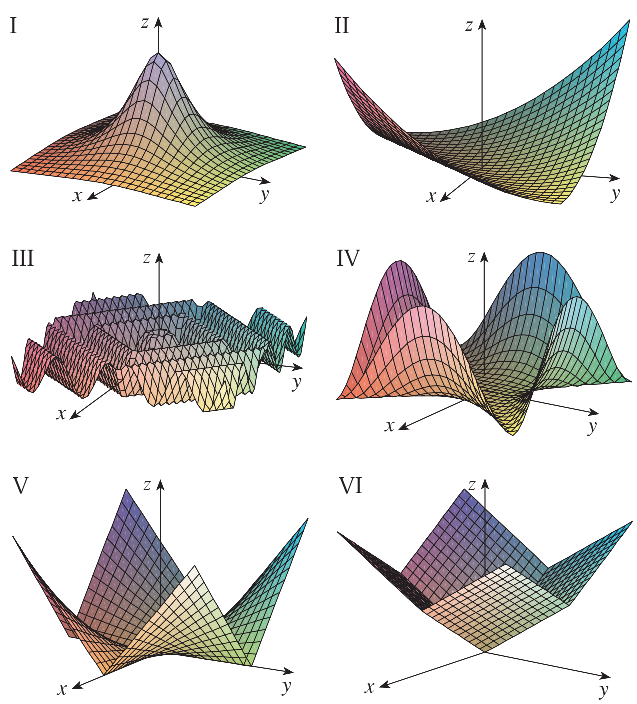
(b) $f(x, y) = |xy|$

(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

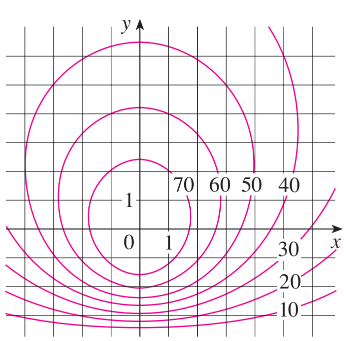
(d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$

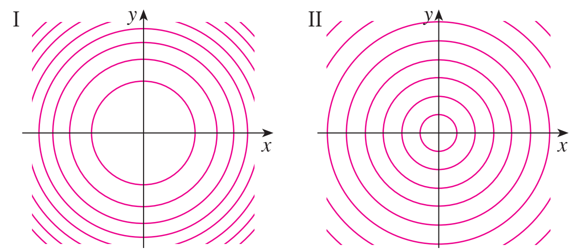
(f) $f(x, y) = \sin(|x| + |y|)$



31. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

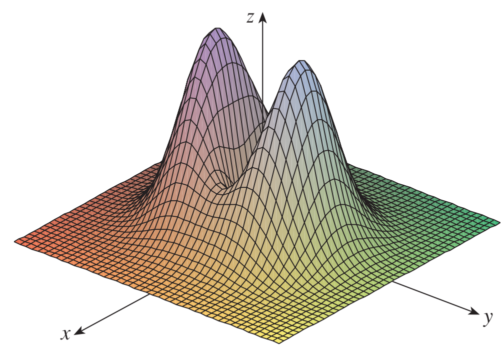


32. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



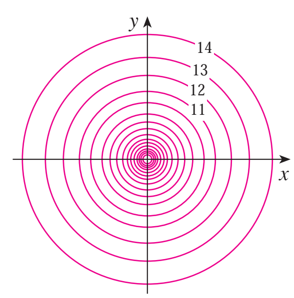
33. Locate the points A and B in the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A? Near B?

34. Make a rough sketch of a contour map for the function whose graph is shown.



35–38 A contour map of a function is shown. Use it to make a rough sketch of the graph of f.

35.



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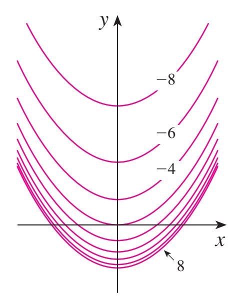

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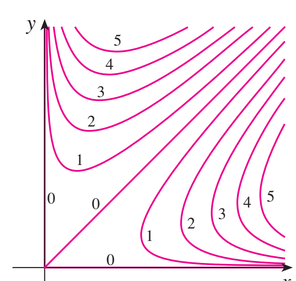

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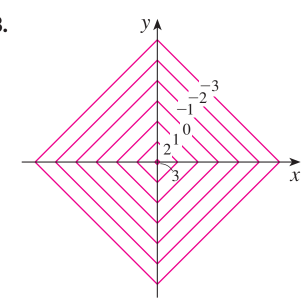


39–46 Draw a contour map of the function showing several level curves.

39. $f(x, y) = (y - 2x)^2$

40. $f(x, y) = x^3 - y$

41. $f(x, y) = y - \ln x$

42. $f(x, y) = e^{y/x}$

43. $f(x, y) = ye^x$

44. $f(x, y) = y \sec x$

45. $f(x, y) = \sqrt{y^2 - x^2}$

46. $f(x, y) = y/(x^2 + y^2)$

47–48 Sketch both a contour map and a graph of the function and compare them.

47. $f(x, y) = x^2 + 9y^2$

48. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

49. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). The level curves of T are called *isothermals* because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2)$$

50. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

51–54 Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

51. $f(x, y) = e^{-x^2} + e^{-2y^2}$

52. $f(x, y) = (1 - 3x^2 + y^2)e^{1-x^2-y^2}$

53. $f(x, y) = xy^2 - x^3$ (monkey saddle)

54. $f(x, y) = xy^3 - yx^3$ (dog saddle)

55–60 Match the function (a) with its graph (labeled A–F on page 869) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

55. $z = \sin(xy)$

56. $z = e^x \cos y$

57. $z = \sin(x - y)$

58. $z = \sin x - \sin y$

59. $z = (1 - x^2)(1 - y^2)$

60. $z = \dfrac{x - y}{1 + x^2 + y^2}$

61–64 Describe the level surfaces of the function.

61. $f(x, y, z) = x + 3y + 5z$

62. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

63. $f(x, y, z) = x^2 - y^2 + z^2$

64. $f(x, y, z) = x^2 - y^2$

65–66 Describe how the graph of g is obtained from the graph of f.

65. (a) $g(x, y) = f(x, y) + 2$ (b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$ (d) $g(x, y) = 2 - f(x, y)$

66. (a) $g(x, y) = f(x - 2, y)$ (b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

67–68 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the "peaks and valleys." Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be "local maximum points"? What about "local minimum points"?

67. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

68. $f(x, y) = xye^{-x^2-y^2}$

69–70 Use a computer to graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

69. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$

70. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

71. Use a computer to investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c?

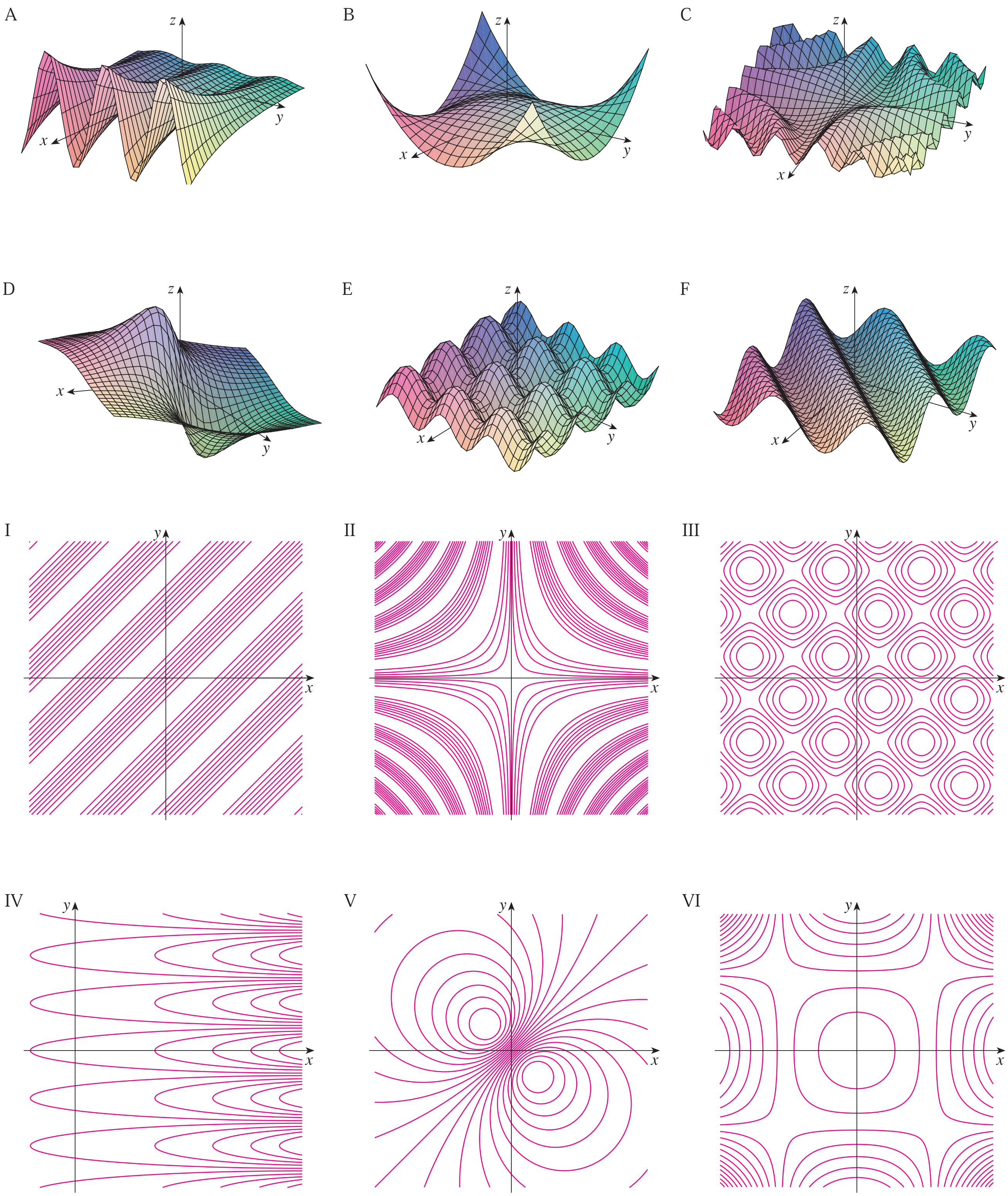
72. Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b?

73. Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

Graphs and Contour Maps for Exercises 55–60



74. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2} \qquad f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2} \qquad f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$$

and
$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$$

obtained from the graph of g?

75. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^{\alpha}K^{1-\alpha}$ can be expressed as

$$\ln\frac{P}{K} = \ln b + \alpha \ln\frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 3) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 LIMITS AND CONTINUITY

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

TABLE 1 Values of $f(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.) It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$